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ERRATUM

# Erratum to: Atomic and molecular decompositions of anisotropic Besov spaces

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We give a corrected proof of Lemma 3.1 in [1].

While the statement of [1, Lemma 3.1] is true, its proof is incorrect. The argument contains a serious defect which can not be easily corrected. The inequality that appears in [1] before (3.5) is not true. If this inequality was true, then we could conclude that, even for a non doubling measure $\mu$, (3.5) was also true. But there exist some non doubling measures for which (3.5) is not true. Since this result plays a fundamental role in the rest of the paper, it becomes compulsory to provide the correct proof of [1, Lemma 3.1].

**Lemma 1.1** *Suppose $K$ is a compact subset of $\mathbb{R}^n$, $0 < p < \infty$, and $\mu$ is a $\rho_A$-doubling measure on $\mathbb{R}^n$ with respect to some expansive dilation $A$. Suppose $f \in \mathcal{S}'(\mathbb{R}^n)$ and $\operatorname{supp} \hat{f} \subset (A^*)^j K$ for some $j \in \mathbb{Z}$. Then*

$$\left( \sum_{k \in \mathbb{Z}^n} \sup_{z \in Q_{j,k}} |f(z)|^p \mu(Q_{j,k}) \right)^{1/p} \leq C ||f||_{L^p(\mu)}, \tag{1.1}$$

*where $Q_{j,k} = A^{-j}([0,1]^n + k)$, and the constant $C = C(K, p, \mu)$ depends on $K$, $p$, and the doubling constant of $\mu$.*

*Proof* We claim that it suffices to show (1.1) only for $j = 0$. Let $f \in \mathcal{S}'(\mathbb{R}^n)$ be such that $\operatorname{supp} \hat{f} \subset (A^*)^j K$ for some $j \in \mathbb{Z}$. Since $f$ is a regular distribution, i.e., $f$ is identified with

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some locally integrable function, we can define a dilate $g \in \mathcal{S}'(\mathbb{R}^n)$ of $f$ by $g(x) = f(A^{-j}x)$. Then, the support of the distribution $\hat{g}$ satisfies

$$\text{supp } \hat{g} = (A^*)^{-j}(\text{supp } \hat{f}) \subset K.$$

Let $\mu_j$ be a dilate of a measure $\mu$ given by $\mu_j(E) = \mu(A^{-j}E)$ for Borel subsets $E \subset \mathbb{R}^n$. Observe that $\mu_j$ has the same doubling constant as $\mu$. Assuming that (1.1) holds for $j = 0$ we have

$$\left( \sum_{k \in \mathbb{Z}^n} \sup_{z \in Q_{0,k}} |g(z)|^p \mu_j(Q_{0,k}) \right)^{1/p} \leq C||g||_{L^p(\mu_j)}. \tag{1.2}$$

Observe that $\mu_j(Q_{0,k}) = \mu(A^{-j}(Q_{0,k})) = \mu(Q_{j,k})$. Moreover,

$$\sup_{z \in Q_{0,k}} |g(z)|^p = \sup_{z \in Q_{0,k}} |f(A^{-j}z)|^p = \sup_{z \in Q_{j,k}} |f(z)|^p.$$

Finally, by the change of variables

$$\int_{\mathbb{R}^n} |g(x)|^p d\mu_j(x) = \int_{\mathbb{R}^n} |f(A^{-j}x)|^p d\mu_j(x) = \int_{\mathbb{R}^n} |f(x)|^p d\mu(x).$$

Combining the above with (1.2) yields (1.1).

To deal with the case $j = 0$ in (1.1) we shall apply [2, Lemma 8.3] which is an adaption of Peetre's mean value inequality [3, Lemma A.4]. Note that the proof of this result in [2] is self-contained and does not depend on any conclusions drawn from [1, Lemma 3.1]. Let $\mathcal{Q} = \{Q_{j,k} : j \in \mathbb{Z}, k \in \mathbb{Z}^n\}$. For $Q = Q_{j,k} \in \mathcal{Q}$ we denote scale$(Q) = -j$ and $x_Q = A^{-j}k$. For any $Q \in \mathcal{Q}$ we define

$$a_Q = \sup_{y \in Q} |f(y)|, \quad b_Q = \sup \left\{ \inf_{y \in P} |f(y)| : \text{scale}(P) = \text{scale}(Q) - \gamma, P \cap Q \neq \emptyset \right\}.$$

For any $r, \lambda > 0$ define a majorant sequence

$$(a^*_{r,\lambda})_Q = \left( \sum_{P \in \mathcal{Q}_0} \frac{|a_P|^r}{(1 + \rho_A(x_Q - x_P))^\lambda} \right)^{1/r}.$$

Likewise we define $(b^*_{r,\lambda})_Q$. Then, [2, Lemma 8.3] says that there exists $\gamma \in \mathbb{N}$ such that

$$(a^*_{r,\lambda})_Q \asymp (b^*_{r,\lambda})_Q \quad \text{for all } \; Q \in \mathcal{Q}_0 := \{Q_{0,k} : k \in \mathbb{Z}^n\}, \tag{1.3}$$

with constants independent of $f$ and $Q$. By [2, (2.6)] we have

$$\mu(Q) \leq C(1 + \rho_A(x_P - x_Q))^\beta \mu(P) \quad \text{for } \; P, Q \in \mathcal{Q}_0, \tag{1.4}$$

where $\beta > 1$ is the doubling constant of $\mu$.

We shall apply (1.3) when $r = p$ and $\lambda > \beta + 1$. By (1.3) and (1.4)

$$\begin{aligned} \sum_{Q \in \mathcal{Q}_0} |a_Q|^p \mu(Q) &\leq C \sum_{Q \in \mathcal{Q}_0} |(b^*_{p,\lambda})_Q|^p \mu(Q) \leq C \sum_{Q \in \mathcal{Q}_0} \sum_{P \in \mathcal{Q}_0} \frac{|b_P|^p}{(1 + \rho_A(x_Q - x_P))^\lambda} \mu(Q) \\ &\leq C \sum_{P \in \mathcal{Q}_0} |b_P|^p \mu(P) \sum_{Q \in \mathcal{Q}_0} \frac{1}{(1 + \rho_A(x_Q - x_P))^{\lambda - \beta}} \leq C \sum_{P \in \mathcal{Q}_0} |b_P|^p \mu(P). \end{aligned} \tag{1.5}$$

In the last step we used the fact that $\sum_{k\in\mathbb{Z}^n}(1+\rho_A(k))^{-1-\varepsilon} < \infty$ for $\varepsilon > 0$. Hence,

$$\begin{aligned}\int_Q |f(x)|^p d\mu(x) &\geq \sum_{P\in\mathcal{Q},\mathrm{scale}(P)=-\gamma} \int_{P\cap Q} |f(x)|^p d\mu(x) \\ &\geq \sum_{P\in\mathcal{Q},\mathrm{scale}(P)=-\gamma} \inf_{z\in P} |f(z)|^p \mu(P\cap Q) \\ &\geq \sum_{P\in\mathcal{Q},\mathrm{scale}(P)=-\gamma} |b_Q|^p \mu(P\cap Q) = |b_Q|^p \mu(Q).\end{aligned}$$

Summing the above over $Q \in \mathcal{Q}_0$ and combining with (1.5) yields

$$\sum_{Q\in\mathcal{Q}_0} |a_Q|^p \mu(Q) \leq C \sum_{Q\in\mathcal{Q}_0} \int_Q |f(x)|^p d\mu(x) = ||f||^p_{L^p(\mu)}.$$

This completes the proof of Lemma 1.1. □

## References


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